

Buoyancy Instabilities in Degenerate, Collisional, Magnetized Plasmas

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ABSTRACT

In low-collisionality plasmas, anisotropic heat conduction due to a magnetic field leads to buoyancy instabilities for any nonzero temperature gradient. We study analogous instabilities in degenerate *collisional* plasmas, i.e., when the electron collision frequency is large compared to the electron cyclotron frequency. Although heat conduction is nearly isotropic in this limit, the small residual anisotropy ensures that collisional degenerate plasmas are also convectively unstable independent of the sign of the temperature gradient. We show that the range of wavelengths that are unstable is independent of the magnetic field strength, while the growth time increases with decreasing magnetic field strength. We discuss the application of these collisional buoyancy instabilities to white dwarfs and neutron stars. Magnetic tension and the low specific heat of a degenerate plasma significantly limit their effectiveness; the most promising venues for growth are in the liquid oceans of young, weakly magnetized neutron stars ($B \lesssim 10^9$ G) and in the cores of young, high magnetic field white dwarfs ($B \sim 10^9$ G).

Key words: convection – instabilities – plasmas – magnetohydrodynamics – white dwarfs – stars: neutron

1 INTRODUCTION

The presence of anisotropic thermal conduction in magnetized low-collisionality plasmas fundamentally changes the Schwarzschild (1958) criterion for convective stability (Balbus 2000, 2001; Quataert 2008, hereafter Q08). Instead of convection being driven solely by an inwardly increasing entropy gradient, low-collisionality plasmas are convectively unstable for both *outwardly* and *inwardly* increasing temperature gradients (Balbus 2000, Q08). These instabilities have been dubbed the magnetothermal instability (MTI; $dT/dr < 0$) and the heat-flux driven buoyancy instability (HBI; $dT/dr > 0$).

The key driver for both of these instabilities is the highly anisotropic heat flux established by even a dynamically weak magnetic field. In particular, if the electron cyclotron frequency is large compared to the electron collision frequency, heat flows almost entirely along magnetic field lines. However, in the presence of a magnetic field, the conductivity tensor *always* possesses some degree of anisotropy, even when collisions are rapid. Thus, any plasma in which electron conduction is energetically important may be susceptible to the MTI and HBI. This motivates us to gen-

eralize previous analyses of conduction-mediated buoyancy instabilities to collisional plasmas. We are particularly interested in the possible application of this physics to the highly conducting plasmas in neutron stars (NSs) and white dwarfs (WDs) and so we will allow for the possibility of degenerate plasmas.

A reasonable measure of the collisionality of a plasma is given by the ratio of the electron cyclotron frequency to the electron collision frequency, $\omega_g \tau$, where $\omega_g = eB/\gamma m_e c$ is the Larmor frequency of electrons with Lorentz factor γ , B is the magnetic field strength, and τ is the time between collisions. To motivate the parameter regime of interest in this paper, we estimate τ as the inverse of the electron-ion collision rate, $\nu_{ei} = n_i \sigma_{ei} v_e$, where n_i is the number density of ions, σ_{ei} is the Coulomb cross section, and v_e is the (thermal or Fermi) velocity of the electrons. For the degenerate plasmas of interest in WDs and NSs, the kinetic energy equals the Fermi energy E_F . Setting the cross section to be $\sigma_{ei} = \pi b^2 \ln \Lambda$, where the impact parameter is $b = Ze^2/E_F$, Z is the ion's charge, and $\ln \Lambda$ is the Coulomb logarithm, which is of order unity in degenerate plasmas (Clayton 1983), we find

$$\omega_g \tau \approx 0.1 \left(\frac{B}{10^8 \text{ G}} \right) \frac{1}{1 + (\rho_6/\mu_e)^{2/3}} \left(\frac{1}{\ln \Lambda} \right) \frac{1}{Z} \quad (1)$$

where the mass density equals $10^6 \rho_6 \text{ g cm}^{-3}$ and μ_e is

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the electron mean molecular weight. The factor of $[1 + (\rho_6/\mu_e)^{2/3}]^{-1}$ in equation (1) accounts for the extension to relativistic degeneracy. Equation (1) shows that degenerate plasmas are highly collisional unless the magnetic field exceeds $\gtrsim 10^9$ G (interiors of WDs; $\rho_6 \sim 1$, $\mu_e = 2$) or $\gtrsim 10^{14}$ G (interiors of NSs; $\rho_6 \sim 10^{8-9}$, $\mu_e \simeq 10$). Hence, magnetically driven buoyancy instabilities, if they exist in weakly magnetized NSs or WDs, operate in the collisional limit studied in this paper.

The remainder of this paper is organized as follows. In §2, we present a linear calculation of buoyancy instabilities in weakly magnetized plasmas allowing for arbitrary collisionality. We present order of magnitude estimates of the growth rates and conditions for instability in the collisional limit. We then discuss the application of these instabilities to NSs and WDs in §3. We summarize and discuss our results in §4.

2 LINEAR CALCULATION

We consider a magnetized fluid with an arbitrary equation of state and arbitrary collisionality. The basic equations governing the plasma are the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (2)$$

the momentum equation,

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{4\pi} (\nabla \times \mathbf{B}) \times \mathbf{B} - \nabla P - \rho \mathbf{g}, \quad (3)$$

the induction equation,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \quad (4)$$

and the energy equation, taken here to be an equation for the entropy of the fluid,

$$\frac{P}{\Gamma_3 - 1} \left[\frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla) \right] \ln P \rho^{-\Gamma_1} = -\nabla \cdot \mathbf{Q}, \quad (5)$$

where ρ is the density, \mathbf{v} is the fluid velocity, \mathbf{B} is the magnetic field, P is the pressure, \mathbf{g} is the local gravitational acceleration, T is the temperature, $\Gamma_1 = (\partial \ln P / \partial \ln \rho)_s$ and $\Gamma_3 - 1 = (\partial \ln T / \partial \ln \rho)_s$ are adiabatic indices, and s is the specific entropy. The heat flux, \mathbf{Q} , is due to radiative diffusion and conduction and is given by

$$\mathbf{Q} = -\chi_{\text{rad}} \nabla T - \chi_{\perp} \left(\nabla - \hat{\mathbf{b}}(\hat{\mathbf{b}} \cdot \nabla) \right) T - \chi_{\parallel} \hat{\mathbf{b}} \left(\hat{\mathbf{b}} \cdot \nabla \right) T, \quad (6)$$

where $\hat{\mathbf{b}} = \mathbf{B}/B$ is the unit vector in the direction of \mathbf{B} and $B = |\mathbf{B}|$. It is helpful to rewrite equation (6) as

$$\mathbf{Q} = -(\chi_{\text{rad}} + \chi_{\text{cond}}) \nabla T - \chi'_{\parallel} \hat{\mathbf{b}} \left(\hat{\mathbf{b}} \cdot \nabla \right) T, \quad (7)$$

where $\chi_{\text{cond}} = \chi_{\perp}$ is the "isotropic" part of the electron conductivity and $\chi'_{\parallel} = \chi_{\parallel} - \chi_{\perp}$ is then the enhancement of the electron conductivity along the magnetic field. From Braginskii (1965),

$$\chi_{\perp} = \frac{\chi}{1 + (\omega_g \tau)^2} \quad \text{and} \quad \chi_{\parallel} = \chi, \quad (8)$$

where χ is the nonmagnetic thermal conductivity. For $\omega_g \tau \gg 1$, the plasma has low collisionality, $\chi_{\perp} \rightarrow 0$, and $\chi'_{\parallel} = \chi_{\parallel} = \chi$. This is typically the case for interstellar and intergalactic plasmas and stellar coronae. This anisotropic

conduction gives rise to the buoyancy instabilities described in the Introduction. For $\omega_g \tau \ll 1$, on the other hand, the plasma is highly collisional. This is typical of stellar interiors, WDs, and weakly magnetized NSs. In this limit, electron conduction is largely isotropic with only a small anisotropic correction: $\chi_{\perp} \simeq \chi$ and $\chi'_{\parallel} \simeq (\omega_g \tau)^2 \chi$.

Finally, we assume that there may be composition gradients in the plasma, but that the timescale for the composition to change is long compared to the other timescales of interest.¹ Thus there is no Lagrangian change in the composition:

$$\frac{\partial \mu_e}{\partial t} + \mathbf{v} \cdot \nabla \mu_e = 0. \quad (9)$$

For concreteness, we have focused on background gradients in the electron mean molecular weight μ_e , but our calculation can easily be generalized to allow for other composition gradients (e.g., in the ion abundance).

We now perform a standard WKBJ analysis on equations (2)-(9). We linearize equations (2) - (9) using the Boussinesq approximation as in Balbus (2000) and Q08. We assume perturbations of the form $\exp(\sigma t - i\mathbf{k} \cdot \mathbf{x})$ and that gravity is in the vertical direction. Under these assumptions the perturbed continuity equation is given by

$$\mathbf{k} \cdot \delta \mathbf{v} = 0, \quad (10)$$

which is the standard incompressibility condition; the perturbed momentum equation is

$$\sigma \delta \mathbf{v} = -i\mathbf{k} \frac{\mathbf{B} \cdot \delta \mathbf{B}}{4\pi \rho} + i\delta \mathbf{B} \frac{\mathbf{B} \cdot \mathbf{k}}{4\pi \rho} - i\mathbf{k} \frac{\delta P}{\rho} + \frac{\delta \rho}{\rho} \frac{\nabla P}{\rho}, \quad (11)$$

the perturbed induction equation is

$$\delta \mathbf{B} = i\delta \mathbf{v} \frac{\mathbf{B} \cdot \mathbf{k}}{\sigma}, \quad (12)$$

and the perturbed heat equation is

$$\frac{P}{\Gamma_3 - 1} \left(-\Gamma_1 \sigma \frac{\delta \rho}{\rho} + \delta \mathbf{v} \cdot \nabla \ln P \rho^{-\Gamma_1} \right) = -i\mathbf{k} \cdot \delta \mathbf{Q} \quad (13)$$

where

$$\delta \mathbf{Q} = -i\chi_{\text{iso}} \mathbf{k} \delta T - \delta \chi_{\text{cond}} \nabla T - \delta \chi'_{\parallel} \hat{\mathbf{b}} \left(\hat{\mathbf{b}} \cdot \nabla T \right) - \chi'_{\parallel} \left[\delta \hat{\mathbf{b}} \left(\hat{\mathbf{b}} \cdot \nabla T \right) + \hat{\mathbf{b}} \left(\delta \hat{\mathbf{b}} \cdot \nabla T \right) \right] - i\chi'_{\parallel} \hat{\mathbf{b}} \left(\mathbf{k} \cdot \hat{\mathbf{b}} \delta T \right), \quad (14)$$

and $\chi_{\text{iso}} = \chi_{\text{rad}} + \chi_{\text{cond}}$ is the sum of the perpendicular electron and radiative conductivities. In the perturbed heat equation, we have taken $\delta P \approx 0$ as our implementation of the Boussinesq approximation.² Note that equation (14) in-

¹ In §3, we focus on the application of the collisional MTI to the $\rho \sim 10^{4-6}$ g cm⁻³ atmospheres of NSs where composition-changing reactions are negligible (excluding episodic fusion due to accretion).

² Taking the dot product of \mathbf{k} with equation (11), it is easy to show that $\delta(P + B^2/8\pi)/P \ll \delta\rho/\rho$ for short wavelength modes; this suggests that we should take $\delta(P + B^2/8\pi) \approx 0$ in the energy equation as our implementation of the Boussinesq approximation. However, this introduces spurious growth and damping terms into the dispersion relation, as we confirmed by taking the incompressible limit, i.e., the high $\beta \equiv P/(B^2/8\pi)$ limit, of the full non-Boussinesq perturbation equations. Thus we implement the Boussinesq approximation by taking $\delta P/P \ll \delta\rho/\rho$ in the energy equation.

cludes the perturbed conductivities $\delta\chi_{\text{cond}}$, and $\delta\chi'_{\parallel}$. We ignore perturbations in the conductivities proportional to δT since these are higher order in $1/kH$, where H is the scale height. However, this leaves perturbations with respect to \mathbf{B}

$$\delta\chi_{\text{cond}} = -\chi'_{\parallel} [1 + (\omega_g\tau)^2]^{-1} \frac{2\hat{\mathbf{b}} \cdot \delta\mathbf{B}}{B}, \quad (15)$$

and $\delta\chi'_{\parallel} = -\delta\chi_{\text{cond}}$. Using equation (15) in the perturbed heat flux, we find

$$\begin{aligned} -i\mathbf{k} \cdot \delta\mathbf{Q} = & -\left[\chi_{\text{iso}}k^2 + \chi'_{\parallel} (\hat{\mathbf{b}} \cdot \mathbf{k})^2\right] \delta T + \\ & i\chi'_{\parallel} \left[(\mathbf{k} \cdot \delta\hat{\mathbf{b}}) (\hat{\mathbf{b}} \cdot \nabla T) + (\mathbf{k} \cdot \hat{\mathbf{b}}) (\delta\hat{\mathbf{b}} \cdot \nabla T)\right] - \\ & i2 \frac{\chi'_{\parallel}}{1 + (\omega_g\tau)^2} \frac{\delta\mathbf{B} \cdot \hat{\mathbf{b}}}{B} \left[\mathbf{k} \cdot \nabla T - (\mathbf{k} \cdot \hat{\mathbf{b}}) (\hat{\mathbf{b}} \cdot \nabla T)\right]. \end{aligned} \quad (16)$$

We calculate δT for any given equation of state, $P(\rho, T, \mu_e)$, using the Boussinesq approximation ($\delta P = 0$) via

$$\delta P = 0 = \left(\frac{\partial P}{\partial \rho}\right)_{T, \mu_e} \delta \rho + \left(\frac{\partial P}{\partial T}\right)_{\rho, \mu_e} \delta T + \left(\frac{\partial P}{\partial \mu_e}\right)_{\rho, T} \delta \mu_e. \quad (17)$$

This implies

$$\frac{\delta T}{T} = \left[\frac{\partial \ln T}{\partial \ln \rho}\right]_{P, \mu_e} \frac{\delta \rho}{\rho} + \left[\frac{\partial \ln T}{\partial \ln \mu_e}\right]_{P, \rho} \frac{\delta \mu_e}{\mu_e}, \quad (18)$$

where

$$\left[\frac{\partial \ln T}{\partial \ln \rho}\right]_{P, \mu_e} = -\frac{\rho}{T} \frac{(\partial P / \partial \rho)_{T, \mu_e}}{(\partial P / \partial T)_{\rho, \mu_e}}, \quad (19)$$

$$\left[\frac{\partial \ln T}{\partial \ln \mu_e}\right]_{P, \rho} = -\frac{\mu_e}{T} \frac{(\partial P / \partial \mu_e)_{T, \rho}}{(\partial P / \partial T)_{\rho, \mu_e}}. \quad (20)$$

The δT perturbation has two components: one due to the density perturbation and one due to the μ_e perturbation. To calculate $\delta\mu_e$, we use the perturbed form of equation (9):

$$\sigma \delta\mu_e = \delta\mathbf{v} \cdot \nabla \mu_e. \quad (21)$$

The temperature perturbation in equation (18) induced by the density perturbation depends on the thermodynamics of the plasma. For instance, equation (19) gives $[\partial \ln T / \partial \ln \rho]_{P, \mu_e} = -1$ for an ideal gas and $\delta\mu_e = 0$. On the other hand, one can show that for a plasma of non-degenerate ions and degenerate electrons, in which the electrons dominate the pressure (e.g., in a WD or in the outer parts of a NS), $[\partial \ln T / \partial \ln \rho]_P = -P_e / n_i k_B T$, where n_i is the number density of ions. The factor of $ZE_F / k_B T$ difference between the ideal gas and the degenerate plasma arises from the fact that the entropy of the degenerate plasma is contained in the ions, which do not contribute significantly to the pressure (Shapiro & Teukolsky 1986). Finally, we note that at very high densities, when the “ions” (protons and neutrons) themselves are degenerate (e.g., in the core of a NS), the heat capacity is reduced by an additional factor of $k_B T / E_{F, \text{ion}}$, where $E_{F, \text{ion}}$ is the ion Fermi energy.

We now combine the perturbed induction equation (eq. [12]), the perturbed momentum equation (eq. [11]), and the incompressibility condition (eq. [10]) to find:

$$[\sigma^2 + (\mathbf{k} \cdot \mathbf{v}_A)^2] \delta\mathbf{v} = -\sigma \left(g - \frac{\mathbf{k} \cdot \mathbf{g}}{k^2} \mathbf{k}\right) \frac{\delta\rho}{\rho}. \quad (22)$$

Combining equations (13), (16), (18), (21), and (22) we arrive at the final form of the dispersion relation:

$$\begin{aligned} & -\sigma\bar{\sigma}^2 - \sigma N^2 \frac{k_{\perp}^2}{k^2} + (\omega_{\text{iso}} + \omega_{\text{cond}, \parallel}) \\ & \times \left(\bar{\sigma}^2 \left[\frac{\partial \ln T}{\partial \ln \rho} \right]_{P, \mu_e} + \left[\frac{\partial \ln T}{\partial \ln \mu_e} \right]_{P, \rho} g \frac{\partial \ln \mu_e}{\partial z} \frac{k_{\perp}^2}{k^2} \right) \frac{P_{\text{Th}}}{P} \\ & - \omega_{\text{cond}, \parallel} g \frac{\partial \ln T}{\partial z} \left[\frac{k_{\perp}^2}{k^2} (1 - 2\Theta b_z^2) + 2\Theta \frac{b_x k_x b_z k_z}{k^2} \right] \frac{P_{\text{Th}}}{P} \\ & - 2 \frac{\omega_{\text{cond}, \parallel}}{1 + (\omega_g\tau)^2} g \frac{\partial \ln T}{\partial z} \left[\frac{k_z^2}{k^2} \frac{b_x k_x}{\mathbf{k} \cdot \hat{\mathbf{b}}} - \frac{k_{\perp}^2}{k^2} \frac{k_z b_z}{\mathbf{k} \cdot \hat{\mathbf{b}}} \right] \frac{P_{\text{Th}}}{P} = 0 \end{aligned} \quad (23)$$

where \mathbf{k}_{\perp} is the component of the \mathbf{k} -vector *perpendicular* to \mathbf{g} , i.e. $k_{\perp}^2 = k_x^2 + k_y^2$,

$$N^2 = g \left(\frac{1}{\Gamma_1} \frac{\partial \ln P}{\partial z} - \frac{\partial \ln \rho}{\partial z} \right)$$

is the Brunt-Väisälä frequency, $\Theta \equiv 1 - (1 + [\omega_g\tau]^2)^{-1}$, and $\bar{\sigma}^2 = \sigma^2 + (\mathbf{k} \cdot \mathbf{v}_A)^2$. We have also defined the characteristic thermal frequencies as

$$\begin{aligned} \omega_{\text{iso}} &= \frac{\Gamma_1}{\Gamma_3 - 1} \chi_{\text{iso}} \frac{T}{P_{\text{Th}}} k^2, \\ \omega_{\text{cond}, \parallel} &= \frac{\Gamma_1}{\Gamma_3 - 1} \chi'_{\parallel} \frac{T}{P_{\text{Th}}} (\mathbf{k} \cdot \hat{\mathbf{b}})^2, \end{aligned}$$

where P_{Th} is the thermal component of the total pressure.

The dispersion relation in equation (23) generalizes the analyses of Balbus (2000, 2001) and Q08 to the case of arbitrary collisionality and degeneracy. We recover Q08's dispersion relation by taking the collisionless limit, i.e., $\omega_g\tau \gg 1$, so that $\Theta \rightarrow 1$, and by specializing to an ideal gas equation of state and no composition gradients.

Motivated by the application to NSs and WDs, we now assume the following hierarchy of timescales: $t_{\text{dyn}} \ll (\mathbf{k} \cdot \mathbf{v}_A)^{-1} \ll \omega_{\text{cond}, \parallel}^{-1}$, and consider the low frequency (small σ) expansion of the dispersion relation in equation (23). The unstable root of the dispersion relation is then given by

$$\sigma \approx \frac{(\omega_{\text{iso}} + \omega_{\text{cond}, \parallel}) [\Omega_{\text{st}}^2 - \Omega_{\text{des}}^2]}{N^2 k_{\perp}^2 / k^2 + (\mathbf{k} \cdot \mathbf{v}_A)^2}, \quad (24)$$

where we have defined the stabilizing term as

$$\Omega_{\text{st}}^2 = \frac{P_{\text{Th}}}{P} \left\{ (\mathbf{k} \cdot \mathbf{v}_A)^2 \left[\frac{\partial \ln T}{\partial \ln \rho} \right]_{P, \mu_e} + g \frac{\partial \ln \mu_e}{\partial z} \frac{k_{\perp}^2}{k^2} \left[\frac{\partial \ln T}{\partial \ln \mu_e} \right]_{P, \rho} \right\}, \quad (25)$$

and the (possibly) destabilizing term as

$$\begin{aligned} \Omega_{\text{des}}^2 \equiv & \frac{\omega_{\text{cond}, \parallel}}{\omega_{\text{iso}} + \omega_{\text{cond}, \parallel}} g \frac{\partial \ln T}{\partial z} \frac{P_{\text{Th}}}{P} \times \\ & \left[\frac{k_{\perp}^2}{k^2} (1 - 2\Theta b_z^2) + 2\Theta \frac{b_z k_z b_x k_x}{k^2} \right. \\ & \left. - \frac{2(\hat{\mathbf{b}} \cdot \mathbf{k})^{-1}}{1 + (\omega_g\tau)^2} \left(b_z k_z \frac{k_{\perp}^2}{k^2} - b_x k_x \frac{k_z^2}{k^2} \right) \right] \end{aligned} \quad (26)$$

For $N^2 > 0$ (i.e., a Schwarzschild stable plasma), equation (24) gives the instability criterion:

$$\Omega_{\text{st}}^2 - \Omega_{\text{des}}^2 > 0. \quad (27)$$

Note that our definition of Ω_{st} is such that $\Omega_{\text{st}}^2 < 0$ since i) $[\partial \ln T / \partial \ln \rho]_{P, \mu_e} < 0$ and ii) $\partial \ln \mu_e / \partial z < 0$ for convective stability (in the sense of the Schwarzschild criterion, $N^2 > 0$). Equation (27) is also the general condition

for instability that one can derive by applying the Routh-Hurwitz criterion as in Balbus (2000). Physically, equation (27) is the condition that the destabilizing thermal effects of anisotropic conduction in the collisional limit exceed the stabilizing effects of magnetic tension and a stable μ_e gradient. The μ_e gradient is particularly stabilizing in a degenerate plasma. First, it increases the magnitude of N^2 , thus suppressing the growth rate (eq. [24]). Even more importantly, the effect of a stabilizing μ_e gradient is at least a factor of ZE_F/k_BT larger than a comparable destabilizing thermal gradient. Thus in a degenerate plasma, instability can only arise when the μ_e gradient is essentially zero. For now, we assume that this is the case. We will return to this question in §4.

Focusing on the case of zero μ_e gradient, we find that the condition for instability is

$$(\mathbf{k} \cdot \mathbf{v}_A)^2 - \Omega_{\text{des}}^2 \left[\frac{\partial \ln T}{\partial \ln \rho} \right]_{P, \mu_e}^{-1} < 0. \quad (28)$$

For a collisional plasma, i.e., $\omega_g \tau \ll 1$, the instability criterion becomes, at the order of magnitude level,

$$(\mathbf{k} \cdot \mathbf{v}_A)^2 \lesssim \omega_g^2 \tau^2 \frac{\chi_{\text{cond}}(\mathbf{k} \cdot \hat{\mathbf{b}})^2}{\chi_{\text{iso}} k^2} \left| g \frac{\partial \ln T}{\partial z} \left[\frac{\partial \ln T}{\partial \ln \rho} \right]_{P, \mu_e}^{-1} \right|. \quad (29)$$

Taking $(\mathbf{k} \cdot \hat{\mathbf{b}})^2 \sim k^2$, we can rewrite equation (29) as

$$(kH)^2 \lesssim \frac{4\pi\rho e^2}{m_e^2 c^2} \tau^2 \frac{\chi_{\text{cond}}}{\chi_{\text{iso}}} g H^2 \left| \frac{\partial \ln T}{\partial z} \left[\frac{\partial \ln T}{\partial \ln \rho} \right]_{P, \mu_e}^{-1} \right|, \quad (30)$$

where H is a typical vertical length-scale in the system, i.e., the pressure scale height. Note that the instability criterion in equation (30) is independent of B because the B^2 that arises from v_A^2 is canceled by the factor of B^2 from ω_g^2 . The order of magnitude growth time for the fastest growing mode can now be determined from equation (24):

$$t_{\text{gr}} \sim \omega_{\text{cond}, \parallel}^{-1} \sim t_{\text{Th}} [(\omega_g \tau)(kH)]^{-2}, \quad (31)$$

where the largest allowed value of kH is given by equation (30) and $t_{\text{Th}} \sim H^2/\chi_{\text{cond}}$ is the local thermal time. Unlike the instability criterion, the growth time does depend on B , with $t_{\text{gr}} \propto B^{-2}$ for a collisional plasma.

Before applying our results to NSs and WDs, we comment on some of the physics of the pure MTI ($b_x = 1$) and HBI ($b_z = 1$) instabilities in the collisional limit. The primary difference in our dispersion relation relative to previous works is the fact that the conductivity depends on B , which introduces terms $\propto \delta\chi$. For the pure HBI case, however, where $b_z = 1$ and $\partial T/\partial z > 0$, we find that all terms associated with $\delta\chi$ exactly cancel. Thus the basic physics of the HBI in Q08 is recovered in the collisional limit, albeit with modifications due to isotropic heat transport associated with ω_{iso} . On the other hand, the pure MTI case, in which $b_x = 1$, has somewhat different physics in the collisional limit. For a low collisionality plasma, the growth rate in the pure MTI limit, for our ordering of timescales, is given by

$$\sigma \approx \omega_{\text{cond}, \parallel} \frac{g|\partial \ln T/\partial \ln z|}{N^2}. \quad (32)$$

By contrast, the growth rate in a collisional plasma is

$$\sigma \approx \omega_{\text{cond}, \parallel} \frac{g|\partial \ln T/\partial \ln z|}{N^2} \left(1 + \frac{2k_z^2}{k_\perp^2} \right). \quad (33)$$

Equation (33) shows that the driving of the MTI is *enhanced* in a collisional plasma relative to what a simple extrapolation of the low-collisionality result might suggest. Namely, we find that including the fact that the magnitude of the anisotropy of the heat flux on B in our calculation increases the growth rate relative to a calculation that considers only a fixed degree of anisotropy (independent of B). Note, however, that, despite the appearance of equation (33), we require $\mathbf{k} \cdot \hat{\mathbf{b}} \neq 0$ for instability because $\omega_{\text{cond}, \parallel} \propto (\mathbf{k} \cdot \hat{\mathbf{b}})^2$; for the pure MTI limit, $b_x = 1$ and thus growth still requires $k_\perp \neq 0$ in a collisional plasma. Thus the additional driving does not change the conditions under which there is growth (e.g., $dT/dz < 0$, weak field, and $k_\perp \neq 0$), only the growth rate.

The enhanced driving of the MTI in a collisional plasma arises because of the dependence of the conductivity on B ; this is analogous to a κ mechanism in stellar oscillation theory (Cox 1983). To illustrate this point, we study the perturbed heat fluxes for the pure MTI case, which are given by

$$\delta \mathbf{Q}_x = -i\chi_{\text{iso}} k_x \delta T - i\chi'_{\parallel} k_x \delta T - k_x \chi'_{\parallel} \xi_z \frac{\partial T}{\partial z}, \quad (34)$$

$$\delta \mathbf{Q}_z = -i\chi_{\text{iso}} k_z \delta T + 2\chi'_{\parallel} k_z \xi_z \frac{\partial T}{\partial z}, \quad (35)$$

where $\xi = \delta v/\sigma$. In the low-collisionality case studied by Q08, the $\delta \mathbf{Q}_x$ term is the same (except for the inclusion of a perturbed isotropic heat flux), while $\delta \mathbf{Q}_z = 0$. The first term on the RHS of equation (35) is the perturbed isotropic diffusion of heat due to the perturbed temperature. The second term is the key new physics: it represents the change in the vertical heat flux due to the perturbed isotropic conductivity that arises from the perturbed magnetic field, i.e., $\delta\chi_{\text{cond}} \propto \delta B$. The physical interpretation is that as a fluid element in an MTI unstable situation is perturbed, the strength of the magnetic field increases. This makes conduction more anisotropic, which increases the conductive heating/cooling of the fluid. This in turn leads to a buoyant response followed by a greater bending of field lines, greater anisotropic conductivity, and yet more heating. This additional driving due to the dependence of the conductivity on B is the origin of the modest enhancement in the growth of the MTI in a collisional plasma.

2.1 The Low-collisionality Limit

For sufficiently strong magnetic fields, $\omega_g \tau \gtrsim 1$, and the low-collisionality limit studied by Balbus (2001) and Quataert (2008) is appropriate, rather than the collisional limit highlighted here. For the applications of interest in this paper, $t_{\text{dyn}} \ll t_{\text{Th}}$, in which case the order of magnitude instability criterion in the low-collisionality limit can be determined from equations (25)-(27) with $\theta \rightarrow 1$. This gives

$$(kH)^2 \lesssim \frac{\chi_{\text{cond}}(\mathbf{k} \cdot \hat{\mathbf{b}})^2}{\chi_{\text{iso}} k^2} \frac{gH^2}{v_A^2} \left| \frac{\partial \ln T}{\partial z} \left[\frac{\partial \ln T}{\partial \ln \rho} \right]_{P, \mu_e}^{-1} \right|. \quad (36)$$

Since equation (36) implies that $k_{\text{max}} \propto B^{-1}$ in the low-collisionality limit, decreasing the magnetic field strength increases the number of unstable modes that can fit in the system. This trend continues until the magnetic field is sufficiently weak that $\omega_g \tau \lesssim 1$ and the collisional limit, rather

than the collisionless limit, is appropriate. In the collisional limit, the magnetic field dependence of the thermal conductivity cancels that of the Alfvén speed and k_{max} is independent of B (eq. [30]). The collisional limit thus allows the largest range of wavelengths to be unstable.

The order of magnitude growth time in the low-collisionality limit is given by

$$t_{gr} \sim t_{Th} (kH)^{-2}. \quad (37)$$

The bound on k from equation (36) implies that the fastest growing mode in the low-collisionality limit has a growth time $t_{gr} \propto B^2$. By contrast, in the collisional limit, the fastest growing mode has $t_{gr} \propto B^{-2}$ (eq. [31]). Thus the growth time of the MTI and/or HBI is minimized when $\omega_g \tau \sim 1$, i.e., at the transition between the collisional and low-collisionality limits.

3 APPLICATIONS TO COMPACT OBJECTS

The interiors of main sequence stars have $\omega_g \tau \ll 1$ and are thus a possible site for the application of the collisional MTI and HBI instabilities. However, it is straightforward to show that the order of magnitude instability condition (eq. [30]) becomes $kH \lesssim 10^{-3}$ in the solar interior; thus no unstable modes can fit in the star. A more promising site for the application of these instabilities is to WDs and NSs, since thermal conduction dominates over radiative diffusion throughout much of the star. Provided that the instability criterion (eq. [30]) can be satisfied and the growth time (eq. [31]) is sufficiently fast, the collisional MTI and HBI may be able to reorient small scale magnetic fields in NSs and WDs or provide a mechanism by which the thermal energy can be tapped to amplify the local magnetic field.

3.1 The Cooling Cores of WDs and NSs

WDs and NSs are born hot, with significant temperature gradients left over from the post-main sequence evolution of their progenitors. After about a thermal time, the core becomes isothermal, with the temperature of the core then decreasing in time as energy is lost through the thermally insulating surface layers. A natural location to study the collisional MTI and HBI is in the cooling core of a WD or NS before it becomes isothermal.³ The condition for instability given by equation (30) can be utilized to estimate the most unstable wavelength; we assume $g \approx GM/R^2$ and $d \ln T/dz \sim R^{-1}$ for simplicity. Using the electron-ion collision time from Yakovlev & Urpin (1980) – which is consistent with equation (1) – we find that the unstable modes in the core of a nonrelativistic degenerate WD must satisfy

$$kR \lesssim 13 Z_6^{-3/2} \ln \Lambda^{-1} M_{0.6}^{2/3} R_9^{-1} T_8^{1/2} \mu_e^{-1/3}, \quad (38)$$

where $T = 10^8 T_8$ K is the initial temperature of the core of the WD, $R = 10^9 R_9$ cm is its radius, and $M = 0.6 M_{0.6} M_\odot$ is its mass.

³ Neutrino cooling at the center of the progenitor star typically leads to a temperature maximum at a non-zero radius in newly formed WDs, so that the core has both $dT/dr < 0$ and $dT/dr > 0$. The WD is thus in principle susceptible to both the collisional MTI and HBI. The same is true for cooling NSs.

In the core of a NS, neutron and proton degeneracy pressure play an important role. The heat capacity of the ions is thus suppressed by a factor of $\sim (k_B T/E_{F,ion})^2$, which reduces the effectiveness of the MTI and HBI relative to WDs. We find that modes are unstable in the core of a NS provided that

$$kR \lesssim 0.2 R_6^{3/2} M_{1.4}^{-1/6} \mu_{e,10}^{5/6} T_8 Z^{-3/2} \ln \Lambda^{-1}. \quad (39)$$

This shows that the cores of weakly magnetized NSs are unlikely to be unstable to the MTI or HBI because the unstable modes have wavelengths larger than the size of the star. This estimate did not include the isotropic conductivity due to neutrons in the core of the NS, which may be substantial (Baiko et al. 2001; Gnedin et al. 2001); this would further inhibit the instability. Strongly magnetized NSs with $B \gtrsim 10^{14}$ G (i.e., magnetars) have $\omega_g \tau \gtrsim 1$ (eq. [1]) and thus the collisional limit in equation (39) does not apply. However, as shown in §2.1 the collisional limit allows the largest range of wavelengths to be unstable. Thus the MTI and HBI are even less likely to be important in the cores of newly formed magnetars.

Equation (38) indicates that a cooling WD is unstable to the collisional MTI and HBI for long wavelengths $\gtrsim 0.1$ R. However, the instability in the core of the WD is driven by the *initial* temperature gradient, which vanishes after one thermal time, $t_{Th} \sim \omega_{cond}^{-1}$, after which the core becomes isothermal. Thus growth is only important if $t_{gr} \lesssim t_{Th}$. Using the estimate of t_{gr} in equation (31), we find the minimum magnetic field required for growth:

$$B \gtrsim 5 \times 10^8 Z_6^{5/2} \ln \Lambda^2 M_{0.6}^{-2/3} R_9 T_8^{-1/2} \mu_{e,2}^{-1/3} \text{ G}. \quad (40)$$

The strongest magnetic fields measured in WDs are $\sim 10^9$ G (Schmidt et al 2003; Vanlandingham et al. 2005). The large magnetic field required for significant growth in the core of a WD (eq. [40]) implies that the MTI and HBI are only potentially important for modifying the magnetic fields of the most strongly magnetized WDs.

3.2 The Collisional MTI in the Oceans of Neutron Stars

Even after the core of a WD/NS has become isothermal, a temperature gradient can persist in the atmosphere of the star. The MTI can operate in this outer envelope. We focus on the application to NSs because our estimates indicate that the growth times exceed the Hubble time in the atmospheres of WDs.

To study the stability properties of NS envelopes we first construct hydrostatic constant-flux equilibrium models of a NS in plane parallel geometry. We focus on weakly magnetized, highly ionized plasmas and assume that the radiative opacity is given by a combination of free-free absorption and Thomson scattering, while the conductive opacity is given by electron-ion scattering (Ventura & Potekhin 2001; Brown et al. 2002; Chang & Bildsten 2003), following Yakovlev & Urpin (1980).⁴ We have also tried more modern conductivities such as those of Potekhin et al. (1999); the

⁴ We have not included magnetic corrections to the opacity in the atmosphere model.

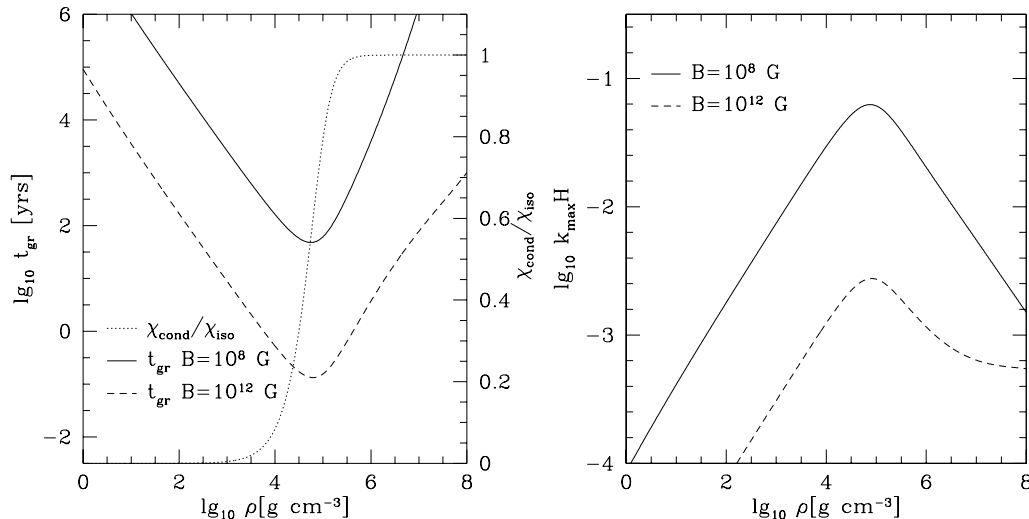


Figure 1. *Left:* Order of magnitude growth time (solid line) for the collisional MTI as a function of density in the envelope of a NS with $T_{\text{eff}} = 10^6$ K and $B = 10^8$ G. The minimum growth time (maximum growth rate) is ≈ 6 yrs and occurs where the opacity transitions from radiative to conductive (dotted line). In the collisional limit, $t_{\text{gr}} \propto B^{-2}$ (eq. [31]). *Right:* The maximum wavevector (minimum wavelength; eq. [30]) relative to the pressure scale height H as a function of ρ for the same model. In the collisional limit, k_{max} is independent of B .

conductivities are consistent to about 30% and the growth times to a factor of ~ 2 , which is sufficient for our purposes.

In the left panel of Figure 1 we plot the order of magnitude growth time for the fastest growing mode, t_{gr} (eq. [31] and [37]), as a function of density in the atmosphere of a $1.4 M_{\odot}$, 10 km NS with $T_{\text{eff}} = 10^6$ K for $B = 10^8$ G (solid line) and $B = 10^{12}$ G (dashed line). Recall that the growth time in the collisional limit scales as $t_{\text{gr}} \propto B^{-2}$, while in the collisionless limit it scales as $t_{\text{gr}} \propto B^2$. The minimum growth time $t_{\text{gr,min}}$ is ~ 50 yrs (~ 0.1 yrs) for $B = 10^8$ G (10^{12} G) and occurs at the “sensitivity strip” (Ventura & Potekhin 2001) where the radiative envelope transitions to the isothermal interior, i.e., where $\chi_{\text{cond}}/\chi_{\text{iso}} \approx 0.5$. This is not surprising since at smaller radii (higher ρ), there is no significant temperature gradient to drive the instability, while at larger radii (lower ρ), the opacity is largely radiative. The Coulomb logarithm Γ is < 173 where the growth peaks, so that the instability occurs largely in the liquid atmosphere, not the crust.

In the right panel of Figure 1, we plot the most unstable wavevector (eq. [30] and [36]) for $B = 10^8$ G and 10^{12} G, respectively. The most unstable wavelength is larger than the scale height by at least a factor of 10 for the collisional case ($B = 10^8$ G) and by a factor of $\gtrsim 300$ for the low collisionality case ($B = 10^{12}$ G). Unlike in the core of the star, this does not completely preclude the MTI from acting because the horizontal wavelength of the mode can be much larger than the vertical wavelength in a thin atmosphere. However, because the tension force is $\propto \mathbf{k} \cdot \mathbf{B}$, Figure 1 does indicate that the instability is most likely to act when the background magnetic field is relatively perpendicular to gravity, so that nearly horizontal perturbations do not significantly bend the magnetic field lines. Moreover, in §2.1 we showed that the most unstable wavelength for the collisional MTI, which is independent of B , is smaller than the most unstable wavelength for the MTI in the low-

collisionality limit. Thus the $B = 10^8$ G result for $k_{\text{max}}H$ in Figure 1 is the shortest wavelength for the MTI in any NS atmosphere. Since $k_{\text{max}}H \lesssim 0.1$ even in this case, magnetic tension severely limits the extent to which the MTI can operate in the atmospheres of NSs. In particular, even if the MTI is present, tension will significantly modify its nonlinear evolution relative to what one would find for $kH \gg 1$: it will be difficult for the field to become relatively radial, as occurs in local simulations of the MTI in the low-collisionality limit (Parrish & Stone 2007).

In spite of the suppression of the MTI by magnetic tension, it is nonetheless useful to study a range of NS atmospheres, parameterized by their effective temperatures T_{eff} , to see how the properties of the instability change in different NSs. We show these results in Table 1 for $B = 10^8$ G, i.e., in the collisional limit. We find that the characteristic instability time is always short ($\lesssim 10^2$ yrs) and that the region of maximum growth is always a liquid ($\Gamma < 173$). For comparison to our estimated growth times, the typical thermal time of a NS at $\sim 10^6$ K is $\approx 10^6$ yrs.⁵ Thus, at a given phase in the evolution of a cooling NS, it is unstable to the MTI on a timescale that is very short compared to the thermal time on which the properties of the NS evolve. The biggest caveat is that, as discussed above, the most unstable wavelength at the location where the growth peaks (eq. [30]) is $k_{\text{max}}H \sim 0.01 - 0.1$ (Table 1); $k_{\text{max}}H$ is the largest, and thus growth is the most likely, for younger, hotter NSs.

4 DISCUSSION

We have studied buoyancy instabilities due to anisotropic heat conduction in collisional, degenerate plasmas, i.e., when

⁵ This is primarily set by neutrino cooling. Once neutrino cooling ceases, the thermal time is $\gtrsim 10^7$ yrs.

Table 1. Table of MTI properties in NS atmospheres with different effective temperatures T_{eff} . The columns include the effective temperature, minimum growth time for the MTI (eq. [31]), the density at which the fastest growth occurs, the most unstable wavelength in units of the local pressure scale height (eq. [30]), and the local Coulomb parameter. Units are K for T_{eff} , yrs for $t_{\text{gr,min}}$ and g cm^{-3} for ρ_{min} . All models are for a $1.4 M_{\odot}$, 10 km NS with $B = 10^8$ G. $t_{\text{gr}} \propto B^{-2}$ in the collisional limit.

T_{eff}	$t_{\text{gr,min}}$	ρ_{min}	$k_{\text{max}}H$	Γ_{min}
$\times 10^5$		$\times 10^4$		
2	549	0.089	0.00653	64.3
4	180	0.540	0.0172	44.0
6	96.9	1.53	0.0302	35.1
8	64.2	3.15	0.0449	29.9
10	47.9	5.53	0.0609	26.3
12	38.5	8.73	0.0779	23.7
14	32.8	12.8	0.0957	21.7
16	29.0	17.3	0.113	20.0
18	26.6	22.8	0.132	18.6
20	24.9	29.1	0.151	17.5

the electron collision frequency is large compared to the electron cyclotron frequency. Although heat conduction is nearly isotropic in this limit, i.e., it is nearly independent of B , the small residual anisotropy drives convective instabilities analogous to the MTI (Balbus 2000) and HBI (Q08) that have been studied previously in low-collisionality plasmas. The physics of the two instabilities is essentially the same in a collisional plasma, although there is additional driving of the MTI due to the magnetic-field dependence of the thermal conductivity (see the discussion near eq. [33]).

In a low-collisionality plasma, the condition for the MTI and/or HBI to grow depends on the magnetic field strength B , with magnetic tension suppressing the instability for larger B (§2.1). The growth time of the instability also depends on B , with the fastest growing mode having $t_{\text{gr}} \propto B^2$. By contrast, in a collisional plasma, the competition between tension and the destabilizing effects of a magnetic-field dependent thermal conductivity leads to an instability criterion that is independent of B (eq. [30]); the growth time in the collisional limit depends on B , however, with $t_{\text{gr}} \propto B^{-2}$. Magnetic tension thus has the smallest effect on the MTI and HBI for marginally collisional plasmas in which the electron collision frequency is comparable to the electron cyclotron frequency.

The collisional MTI and HBI can in principle operate in the cores of young WDs and NSs and in the outer envelopes of NSs (§3). However, magnetic tension and the low specific heat of a degenerate plasma severely limit the importance of the MTI and HBI in these environments. The MTI and HBI can undergo several e-foldings in the core of a young, high magnetic field WD ($\gtrsim 5 \times 10^8$ G), before thermal conduction wipes out the temperature gradient that drives the instability in the first place (eq. [40]); however, only a small fraction of WDs have magnetic fields this strong (Vanlandingham et al. 2005). We find that there is very unlikely to be analogous growth of the MTI or HBI in the cores

of young NSs (eq. [39]). However, the MTI may be present in the liquid oceans of NSs (see Fig. 1), where a finite temperature gradient persists even after the core has become isothermal. The most promising candidates are young (hot), weakly magnetized ($B \lesssim 10^9$ G) NSs. Even in this case, however, the fastest growing modes have $k_{\text{max}}H \sim 0.1$ and are thus restricted to very long wavelengths (Table 1). For more strongly magnetized NSs, in which conduction is more anisotropic, growth in the outer atmosphere is *less likely* because of the increased stabilization by magnetic tension.

Given these results, one interesting possibility is that the collisional MTI could contribute to amplifying and/or modifying the magnetic fields in the atmospheres of weakly magnetized NSs such as millisecond pulsars or NSs in low-mass x-ray binaries. In these systems, it is believed that an initially strong birth magnetic field either decayed away or was buried by accretion, leaving the NS with a weak field that is potentially unstable to the MTI.

However, it is unclear whether the collisional MTI will grow significantly in this context and, if it does, how it will saturate. In weak-field simulations of the MTI in low-collisionality plasmas, the magnetic field is amplified by a factor of $\sim 10 - 30$ and the instability saturates by re-orienting the magnetic field to be largely in the direction of the background gravitational field (Parrish & Stone 2005, 2007). In the oceans of NSs, however, the instability is restricted to very long wavelengths, larger than the local pressure scale height of the atmosphere (Fig. 1 and Table 1). Thus the instability is most likely to act in regions where the local magnetic field is relatively perpendicular to gravity, so that modes with long horizontal wavelengths do not significantly bend the magnetic field. Given these stringent requirements for growth, it is difficult to see how the MTI can significantly change the structure of the NSs magnetic field. Nonlinear simulations are required to assess this in detail and to determine if the additional driving due to the magnetic field-dependent thermal conductivity modifies the nonlinear evolution of the MTI in collisional plasmas.

A background gradient in the composition, in particular in the electron mean molecular weight μ_e , can also act to stabilize the MTI and does so particularly effectively in a degenerate plasma (see the discussion below eq. [27]). In the part of a NSs atmosphere where the instability is the most powerful, however, we do not expect a significant μ_e gradient. The diffusion time through this region is incredibly fast, $\sim 0.1 - 10$ s at $\rho \sim 10^5 \text{ g cm}^{-3}$ (Chang & Bildsten 2004). Thus a significant μ_e gradient is only likely if the growth happens to occur near a compositional discontinuity in the NS. This is not impossible but is unlikely to generically be the case.

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